A theorem in algebra guarantees that it is always possible to do this. We explain the details for the four cases that occur.

**CASE I ■ The denominator $Q(x)$ is a product of distinct linear factors.**

This means that we can write

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that

**2** $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

These constants can be determined as in the following example.

**EXAMPLE 2** Evaluate $\displaystyle\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$.

SOLUTION Since the degree of the numerator is less than the degree of the denominator, we don't need to divide. We factor the denominator as

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$$

Since the denominator has three distinct linear factors, the partial fraction decomposition of the integrand (2) has the form

**3** $$\frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$$

To determine the values of $A$, $B$, and $C$, we multiply both sides of this equation by the product of the denominators, $x(2x - 1)(x + 2)$, obtaining

**4** $$x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$$

Expanding the right side of Equation 4 and writing it in the standard form for polynomials, we get

**5** $$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

The polynomials in Equation 5 are identical, so their coefficients must be equal. The coefficient of $x^2$ on the right side, $2A + B + 2C$, must equal the coefficient of $x^2$ on the left side—namely, 1. Likewise, the coefficients of $x$ are equal and the constant terms are equal. This gives the following system of equations for $A$, $B$, and $C$:

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A \qquad\qquad &= -1 \end{aligned}$$

Solving, we get $A = \frac{1}{2}$, $B = \frac{1}{5}$, and $C = -\frac{1}{10}$, and so

■ We could check our work by taking the terms to a common denominator and adding them.

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx = \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x - 1} - \frac{1}{10}\frac{1}{x + 2}\right) dx$$

$$= \tfrac{1}{2}\ln|x| + \tfrac{1}{10}\ln|2x - 1| - \tfrac{1}{10}\ln|x + 2| + K$$

■ Figure 1 shows the graphs of the integrand in Example 2 and its indefinite integral (with $K = 0$). Which is which?

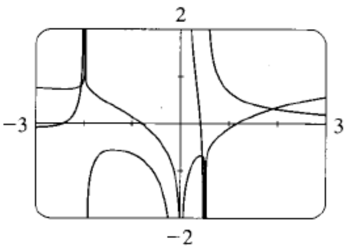


FIGURE 1

In integrating the middle term we have made the mental substitution $u = 2x - 1$, gives $du = 2\,dx$ and $dx = du/2$.

NOTE We can use an alternative method to find the coefficients $A$, $B$, an Example 2. Equation 4 is an identity; it is true for every value of $x$. Let's choose va $x$ that simplify the equation. If we put $x = 0$ in Equation 4, then the second and thir on the right side vanish and the equation then becomes $-2A = -1$, or $A = \frac{1}{2}$. Li $x = \frac{1}{2}$ gives $5B/4 = \frac{1}{4}$ and $x = -2$ gives $10C = -1$, so $B = \frac{1}{5}$ and $C = -\frac{1}{10}$. (You may that Equation 3 is not valid for $x = 0, \frac{1}{2}$, or $-2$, so why should Equation 4 be valid fo values? In fact, Equation 4 is true for all values of $x$, even $x = 0, \frac{1}{2}$, and $-2$. See Exer for the reason.)

**EXAMPLE 3** Find $\displaystyle\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

SOLUTION The method of partial fractions gives

$$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{A}{x - a} + \frac{B}{x + a}$$

and therefore

$$A(x + a) + B(x - a) = 1$$

Using the method of the preceding note, we put $x = a$ in this equation and get $A(2a) = 1$, so $A = 1/(2a)$. If we put $x = -a$, we get $B(-2a) = 1$, so $B = -1/(2$ Thus

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int \left(\frac{1}{x - a} - \frac{1}{x + a}\right) dx$$

$$= \frac{1}{2a}\left(\ln|x - a| - \ln|x + a|\right) + C$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

6
$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + C$$

See Exercises 55–56 for ways of using Formula 6.

**CASE II** ■ ***Q*(*x*) is a product of linear factors, some of which are repeated.**

Suppose the first linear factor $(a_1x + b_1)$ is repeated $r$ times; that is, $(a_1x + b_1)^r$ oc the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equatio