**References**

Murray Bourne http://intmath.com
Paul's Notes: http://tutorial.math.lamar.edu/ DE_complete.pdf
MAS164 Textbook → College Mathematics by S.T.Tan 6th edition
MAS182 Textbook → Calculus with Applications by Lial Greenwell Ritchey
MAS161 Text book → Calculus 6E by James Stewart
Linear algebra for Calculus – by K.Heuvers
MAS208 Textbook → Differential Equations – by Dennis G. Zill
ENG267 Textbook → Process Dynamics Modeling and Control by Babatunde A. Ogunnake and W. Harman Ray

**Vector definition between 2 points**

$$\vec{v}=(x_2, y_2, z_2)-(x_1, y_1, z_1)$$

**Equation for a line in 3D** pg 830 Stew

$\vec{r}=\vec{r}_0+t\,\vec{v}$ where

$\vec{r}_0$ denotes a point on the line.
t Denotes a variable to trace out the line.
$\vec{v}$ Denotes the above vector definition between 2 points

**Distance formula in 3 dimensions** - pg 803 Stew

$$|P_1 P_2|=\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2+(Z_2-Z_1)^2}$$

**Equation for a sphere** - pg 804 Stew

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

**Dot Product** -pg 815 Stew

$$a\cdot b=a_1 b_1+a_2 b_2+a_3 b_3$$

The resultant is a real number (ie scalar)

**THEOREM** - Angle between the vectors a and b – pg 816 Stew

$$a\cdot b=|a||b|\cos\theta$$

**Perpendicular (orthogonal) angles** pg 817 Stew

If $a\cdot b=0$ then the two lines are perpendicular.

**COROLLARY** – If θ is the angle between the nonzero vectors a and b, then

$$\cos(\theta)=\frac{a\cdot b}{|a||b|}$$

**Cross Product** pg822 Stew
The resultant is a vector which is perpendicular to both the other vectors.
where $a=\langle a_1, a_2, a_3\rangle$ $b=\langle b_1, b_2, b_3\rangle$
the cross product is

$$a\times b=\langle a_2 b_3-a_3 b_2, a_3 b_1-a_1 b_3, a_1 b_2-a_2 b_1,\rangle$$

**Orthogonal** – 2 lines are orthogonal if they are perpendicular.
Pg 824 stew
The vector $a\times b$ is orthogonal to both a and b.

**Parallel vectors** Pg 825 Stew
2 vectors are parallel if the cross product **a x b** = 0

**Area of a parallelogram** Pg 825 Stew
The length of a cross product **a x b** is equal to the area of the parallelogram determined by **a** and **b**.

**Equation of a plane** pg 834Stew

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

where **n** = (a,b,c) = normal vector = cross product of 2 vectors on plane and

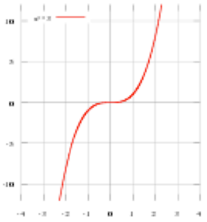
$r_0=(x_{0,}\, y_{0,}\, z_0)$ which is a point on the plane.

**Rolle's Theorem** pg 214 Stew **and Mean Value Theorem** also pg 11 of the first lecture notes.calcintro.pdf
There is a value c in (a,b) such that

$$f'(c)=\left[\frac{f(b)-f(a)}{b-a}\right]$$

Use this theorem to find the gradient (differentiation) at point c which lies half way between a and b. To check it you then differentiate the original formula the normal way and substitute $f'(c)$ back into it.

**Inflection Points** pg752 Tan
The point where a curve changes inflection.



**Curve sketching**
pg 243 stew
or pg 766 Tan

**Laplace Transforms**
How to solve DE's using LT's – pg 78 Ogunnake – 3 step procedure.

$$L\{y'\}=sY(s)-y(0)$$
$$L\{y''\}=s^2 Y(s)-sy(0)-y'(0)$$

∴ the above are our s domain transforms for first and second differentials.
The Best examples of these are in pauls online notes pgs215 – 221. in the document DE_Complete.pdf
**σ** = a dummy time vairiable which gets integrated out. Ogun pg111

**Laplace Manipulation**
When you have multiple transforms added together like :

$$T(s)=\frac{4s+5}{(S^2+4s+6/s)s(s+1)}+\frac{3s+2}{(S^2+2s+3)s(s+1)}+\ldots$$

each one between the +/- signs can be treated individualy.
In the above both terms need partial fracitonal expansion, but it would be advisable to do them individually.

**Transform Domain Model** Ogun p99
**First order system – Dynamic** : General behaviour
**Step function** Ogun p80 known as H(t) or heavyside.
A= size of the step function.
**Inverse Laplace** Ogun pg125
**Transfer Function Table** Ogun p131
**First Order DE transfer function** Ogun p140

K = steady state Gain = $\dfrac{y(\infty)}{A}$ Ogun.p145 or ENG267-11

$\tau$ = Time constant = 0.632AK = $\dfrac{AK}{\sigma}=\dfrac{y(\infty)}{\sigma}$

When t = 3.9 $\tau$ then y(t) is at 98%
When t = 4.6 $\tau$ then y(t) is at 99%
AK reaches 100% in exactly $\tau$ time units. Ogun.pg144
σ = a dummy time vairiable which gets integrated out. Ogun pg111 =

$\sigma=\dfrac{y(\infty)}{\tau}=\dfrac{AK}{\tau}$ which also equals the angle of the slope response at y(t).

Single pole at s =-1/ $\tau$ –> What is a pole? → Poles are on the bottom of a transform and roots are at the top.
Poles and roots are just naming conventions to label the roots of the numerator and roots of the denominator.

**First order system**: Ogun p139 or Lecture notes eng267-11
**Finding Tau and the Gain (K)**

$$a_1\frac{dy}{dt}+a_0 y=bu(t)$$

Let $\tau=\dfrac{a_1}{a_0}$ and $K=\dfrac{b}{a_0}$

$$\rightarrow\quad \tau\frac{dy}{dt}+y=Ku(t)$$

**Step response Model** Ogun p143
**Impulse response Model** Ogun p111
**Second Order Systems** Ogun pg183

$$y(s)=\frac{K}{\tau^2 s^2+2\zeta\tau s+1}u(s)$$

$\zeta<1$ Under-damped Response
$\zeta=1$ Critically damped Response
$\zeta>1$ Over-damped Response

In Simulink this is represented by :-

$$\frac{1}{as^2+bs+1}$$

In Simulink the idea is to organise a and b to get your damped response.
Where :-

$a=\tau^2$ and $b=2\zeta\tau$

Simulink uses State space modelling. Ogun Pg 182
**What is the FVT????** → Final Value Theorem
As shown in Test one 2007 question 1(v)

**Partial Fractional Expansions** best example and explination → pg511 Stew
*General method.*
**1.** Break up polynomial on bottom into it's individual roots (poles). - If we can not because it is an answer of complex form we need to use the procedure called 'Completing the square'
**2.** Above each pole in the position of the numerator write A,B,C......etc.
**3.** Each of these terms are now in the form..

$$\frac{x^2+2x-1}{x(2x-1)(x+2)}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

**4.** Create the common denominator for additions.

$$x^2+2x-1=A(2x-1)(x+2)+B(x)(x+2)+C(x)(2x-1)$$

Then mash it around to look like :

$$x^2+2x-1=(2A+B+2C)x^2+(3A+2B-C)x-2A$$

From here you have 3 simultaneous eqations

$$2A+B+2C=1$$
$$3A+2B-C=2$$
$$-2A \quad .=-1$$

**5.** From here we have 2 different ways we can solve this.
a. By using an inverse matrix solution or
b. by substituting in a number for x to cancel out either A,B or C. This will need to be done 3 times.

**NB.** *Can only be done if the degree in the numerator is smaller than the degree in the denominator.*

**Integration by parts** 161T-7

**Implicit Differentiation** see 161W2-3 Workshop2

**Rule from MAS208 Questions** $(a\times f(x))'=Af'(x)$

**Equilibrium Points and Stability** pg 93 Pauls DE
**1.** Asymptotically stable equilibrium solutions are where points near an equilibrium point move towards it.
**2.** Unstable equilibrium solutions move away from the equilibrium point.
**3.** Semi stable solutions move towards on one side and away on the other.

**Dependant Variable** y is usually what changes with respect to the **independent Variable** which is what we manipulate. Pg11 Stew.

**Differential Equations**

p570 **General Solution** of $\frac{dy}{dx}=f(x)$ is

$\int\frac{dy}{dx}dx=\int f(x)dx$ which is $y=\int f(x)dx$

- Because an integral reverses a derivative.
- and you have to integrate both sides of the equation.

p571 **Particular Solution** is when you find the value of C by substituting in point values from the function line.

**P572 Separable Differential Equations –** Means to be able to get one variable on one side like you normally try to do in an algebraic equation.

$\frac{dy}{dx}=\frac{f(x)}{g(y)}$ is resolved by cross multiplying

$g(y)dy=f(x)dx$ then integrating both sides

$$\int g(y)dy=\int f(x)dx$$

thus $G(y)=F(x)$ where the G an F represent the anti-derivatives of g and f.

**Differential – First Order**
P584 - Write these rules.Finding the integrating factor
**Integrating Factor**
Thr Function $I(x)=e^{\int P(x)dx}$ is called the integrating factor for the differential equation. $\frac{dy}{dx}+P(x)y=Q(x)$

P584 **Solving a Linear First Order Differential Equation** Pauls nots DE_complete.pdf Pg21

1. Put the equation in the linear form $\frac{dy}{dx}+P(x)y=Q(x)$
2. Find the integrating factor $I(x)=e^{\int P(x)dx}$
3. Multiply each term of the equation from Step 1. by $I(x)$ .
4. Replace the sum of the terms on the left with $(I(x)y)'$ .This is because with the interating factor they are nothing more than the product rule.
5. Integrate both sides of the equation.
6. Solve for y.

**Autonomous DE** : Where the independent variable is missing from the RHS. P411stew or p37 Zill

eg : $\frac{dy}{dx}=1+y^2$

**2nd Order DE's – Homogenous** $y_H$ = homogeneous soln.
A second order DE is one that has a 2nd derivative in it defined by either y'' or $\frac{dy^2}{d^2x}$

1. Try to rearrange to the format y''-y'+y=0
2. Let $y=e^{\lambda t}$ and $y'=\lambda e^{\lambda t}$ and $y''=\lambda^2e^{\lambda t}$
3. Substitute the above values in.
4. Rearrange to be $e^{\lambda t}(\lambda^2-\lambda+1)=0$
5. Calculated the inside of the brackets as a quadratic.
**a.** Two real roots $\lambda_1\lambda_2$

$$y_H=Ae^{\lambda_1 x}+Be^{\lambda_2 x}$$

**b.** Complex conjugate roots $\lambda=\alpha\pm i\beta$

$$y_H=e^{\alpha x}(C\cos\beta x+D\sin\beta x)$$

Then insert your initial values to find C & D.
You may have to take the derivative of the answer to find C or D.
**c.** Single real root $\lambda$

$$y_H=(Ax+B)e^{\lambda x}$$

**2nd Order DE's – NON Homogenous**
**1.0** Perform procedure on LHS to get $y_H$ as abve.
$y_p$ = means the particular soln.
**1.1** Use the guessing table on page208L-25

| | |
|---|---|
| poly. Of order n | $y_p=c_1x^n+c_2c^{n-1}+\ldots+c_n$ |
| exp. $e^{\beta x}$ | $y_p=ce^{\beta x}$ |
| Sine or cosine | $y_p=C\sin\beta x+D\cos\beta x$ |
| Soln. to homog. eqn. | $y_p$ = x * (Soln. To Homog eqn.) |
| Constant (ie = 45) | $y_p=C$ |

**2.** For which ever one of these you choose then find y, y' and y''.of it.
**3.** Substitute these derivatives into the original equation and replace the RHS with the item from the guesing table..
**4.** Hard to explain here without an example but rearrange into simultaneous equations to find C and D.
**5.** Insert C and D into $y_p$ from the guessing table.
**6.** General Soln $y=y_H+y_p$
**7.** Now apply initial conditions and find A and B in the $y_H$

**Eulers Methd** pg35 Ha's notes -ODE's
With Eulers method we are trying to get an approximate solution for:

$$\frac{dy}{dt}=yt$$

The approximate numerical equivalent to the differential $\frac{dy}{dt}$ equation is called a difference equation as shown below.

$$\frac{dy}{dt}\approx\frac{y_{k+1}-y_k}{\Delta t}=y_kt_k,\ k=0,1,2,3,\ldots$$

*k* is the point in time. ie 0,1,2,3.. seconds
*Δt* is the time incriment. Ie 0.5seconds or ½ a second.

Rearrange to $y_{k+1}=y_k+\Delta t(y_k k\Delta t)$
Note here that $t_k=k\Delta t$ and *Δt=0.2 seconds if a smaller time step is required.*

**Newtons Method** pg269 Stew

**1.** Plot the graph and find a point close to the intersect to get a starting point which will be $x_1$

**2.** Rearrange to the standard form. Ie $x^6 - 2 = 0$

**3.** Apply $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

**4.** Apply over and over to get a series of numbers $x_1, x_2, x_3, x_4, \ldots$ When 2 of these terms are the same to 8dp then you have your answer.

**Completing the Square**
Combination of Ha's printed handout on integration – pg21
Pauls notes - Alg_Complete.pdf – page 93
Best to look at the examples in Pauls notes here but this is a brief description of the format.

$x^2 + bx = x$ Is the general format of a quadratic.

Add $\left(\frac{b}{2}\right)^2$ to get a factorable quadratic.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

**Sequences**
To find if a sequence is convergant or divergent :

$$a_n = \left(\frac{1 + n - 3n^2}{7 + 4n^2}\right)$$

**1.** Divide by the highest power of n. ie $n^2$ and take the limit

$$\lim_{n\to\infty} a = \left(\frac{\frac{1}{n^2} + \frac{1}{n} - 3}{\frac{7}{n^2} + 4}\right) \rightarrow \lim_{n\to\infty} a = \left(\frac{0 + 0 - 3}{0 + 4}\right)$$

$\rightarrow \lim_{n\to\infty} a = \left(\frac{-3}{4}\right)$ Check this rule with Ha?

***Series*** (are a sum of sequences or functions)

**Tests of Convergence** Ha's hand written notes 6.3
**1.** Integral Test *(Also used to estimate the sum of a series)*
Basically just do a definite integral for the series.
**2.** Comparison Test
**3.** Limit Comparison Test
**4.** Alternating Series Test
**5.** Ratio Test. *Ha's typed notes 'Seq. & Series' pg 52*

Find $\lim_{n\to\infty} . \left(\frac{a_{(n+1)}}{a_n}\right)$ So literally put $a_{(n+1)}$ over $a_n$ and algebraically solve it.

**NB : You will have to divide through by $n^x$ where x is the highest power.**

$\lim_{n\to\infty} . \left|\frac{a_{(n+1)}}{a_n}\right| < 1$ Absolutely convergent

$\lim_{n\to\infty} . \left|\frac{a_{(n+1)}}{a_n}\right| > 1 \text{ or} \rightarrow \infty$ Divergent.

$\lim_{n\to\infty} . \left|\frac{a_{(n+1)}}{a_n}\right| = 1$ Indeterminate → use another method

**6.** Root Test (not needed for MAS161)

**Power Series** pg 759 Stew
**1.** A power series may converge or diverge for some values of x
**2.** The sum of the series is a function where its domain is the set of x where there is convergence.

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a) + \ldots..$$

**3.** This is a power series about a or centred about a.

**4.** First do the Ratio test on the series at hand.
**5.** Which should algebraically reduce to the form (x – a).
**6.** From the Ratio rule we know that for the series to be convergent then -1 < x < 1.
So say you get (x – 3) as n → ∞ :. in this case
-1 < |x – 3|. < 1
So just mash x around until it fits this inequality.
**7.** Then we need to find out whether it converges for 1 and -1 so plug the right numbers in for x an check it.

**Taylor Series** Paul's notes DE_complete.pdf – pg 327 gives a good explanation.
**1.** Take the number of derivatives you need according to the order required. (The order is the power. So if order 4 the do up to the 4th derivative.)
**2.** The requested Taylor series will be asking for an answer about a number. Say x=0. So solve the original equation and all of the derivatives you've made at x=0 in this case..
**3.** Note that the original equation is the 0th derivative and this is used in the initial terms.
**4.** Take these answers and insert them into the Taylor formula.

$$f(x) = \frac{\sum_{n=0}^{\infty} f^n(x_0)}{n!}(x - x_0)^n$$

This is a bit confusing but
$f^n$ is just the result you got at a specific derivative in step **2.**
The $x_0$ is just the number your solving for. Ie : x=0
$(x - x_0)^n$ Here if you were solving for x=5 and it was step 3 then you would leave it at . See Paul's notes for more detail.

**L'Hospital's Rule** Ha's printed notes 'Opening remarks' pg14
When we are taking the limits (ie where a curve crosses another line) we often get to the point where we can't get an answer because we get $\frac{0}{0}$ at a limit.

Eg : $\lim_{x\to 2} a = \left(\frac{x^2 - 4}{x - 2}\right) = \left(\frac{0}{0}\right)$

To resolve this we take the differential of both the numerator and the denominator.

$$\lim_{x\to 2} a = \left(\frac{2x}{1}\right) = 4$$

**Angles** using a 3,4,5 triangle.

| **Degrees** | **0°** | **30°** | **45°** | **60°** | **90°** |
|---|---|---|---|---|---|
| **Radians** | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
| **Sin θ** | $\frac{\sqrt{0}}{2}$ | $\frac{\sqrt{1}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{4}}{2}$ |
| **Cos θ** | $\frac{\sqrt{4}}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{1}}{2}$ | $\frac{\sqrt{0}}{2}$ |
| **Tan θ** | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | **DN θ** |

**Finite Difference Numerical scheme for Partial Differential Equations.**
See pdf that I have to scan into the formula folder or
MAS208 Assignment 4 Q7
Page 518 Zill
j = vertical dimension
I = horizontanl dimension

**Seperation of Variables for multivariable functions** – pg 434 Zill.
Copy lecture notes pg 208L-66

**Solution to Systems of ODE's** – like in Loktra Volterra prey/ predator models.
Lecture notes 208L-57
Zill pg 320
Uses EigenVectors to solve and comes out with a general solution of : -
Case 1 : Lecture notes 208L-57
Case 2 : Lecture notes 208L-58
Case 3 : Lecture notes 208L-58

**When finding Extrema of multi-variable functions**
– like single variable
- set each derivative to =0
- factor out each variable
- find all values of each variable for when each derivative =0
- treat as simultaneous equations to make easier.

**To check for Saddle Point** – Take det. At each point of extrema

$$Det(a,b)\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^{2}$$ If Det(a,b) < 0

Saddle